- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
$$\int_0^1 x\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)dx.$$

Solution 2 by Arkady Alt, San Jose, CA

Solution A.

Let $I=\int\limits_0^1 x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)dx.$

Then $4I=\int\limits_0^1 x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)^2\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)^2 dx=\int\limits_0^1 xu(x)\,v(x)\,dx,$

where $u(x)=\ln\left(2+2\sqrt{1-x^2}\right), v(x)=\ln\left(2-2\sqrt{1-x^2}\right).$

Since $u(x)+v(x)=\ln\left(4x^2\right)=2\ln(2x)$ then

$u^2(x)+v^2(x)+2u(x)\,v(x)=4\ln^2(2x) \iff u(x)\,v(x)=2\ln^2(2x)-\dfrac{u^2(x)+v^2(x)}{2}$

and, therefore, $4I=2\int\limits_0^1 x\ln^2(2x)\,dx-\dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx+\int\limits_0^1 xv^2(x)\,dx\right).$

1. Using substitution and integration by parts we obtain

$2\int\limits_0^1 x\ln^2(2x)\,dx=[t=2x;dt=2dx]=\dfrac{1}{2}\int\limits_0^2 t\ln^2(t)\,dt=\ln^2 2-\dfrac{1}{2}\int\limits_0^2 t\ln t\,dt=$

$\ln^2 2-\ln 2+\dfrac{1}{2}.$

2. Let $t=2+2\sqrt{1-x^2}$. Since $xdx=-\dfrac{(t-2)\,dt}{4}$ then

$\int\limits_0^1 xu^2(x)\,dx=\dfrac{1}{4}\int\limits_4^2-(t-2)\ln^2 tdt=\dfrac{1}{4}\int\limits_2^4(t-2)\ln^2 tdt.$

3. Let $t=2-2\sqrt{1-x^2}$. Since $xdx=\dfrac{(2-t)\,dt}{4}$ then

$\int\limits_0^1 xv^2(x)\,dx=\dfrac{1}{4}\int\limits_0^2(2-t)\ln^2 tdt=-\dfrac{1}{4}\int\limits_0^2(t-2)\ln^2 tdt.$

Hence $\dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx+\int\limits_0^1 xv^2(x)\,dx\right)=\dfrac{1}{8}\left(\int\limits_2^4(t-2)\ln^2 tdt-\int\limits_0^2(t-2)\ln^2 tdt\right)=$

$\dfrac{1}{8}\left(\int\limits_0^4(t-2)\ln^2 tdt-2\int\limits_0^2(t-2)\ln^2 tdt\right).$

Using integration by parts twice we obtain

$$\int (t-2)\ln^2 t dt = \left[\begin{array}{c} p' = t-2; p = \frac{t^2}{2} - 2t \\ q = \ln^2 t; q' = \frac{2\ln t}{t} \end{array}\right] = \left(\frac{t^2}{2} - 2t\right)\ln^2 t - \int (t-4)\ln t dt =$$
$$\left(\frac{t^2}{2} - 2t\right)\ln^2 t - \left(\frac{t^2}{2} - 4t\right)\ln t + \frac{t^2}{4} - 4t.$$

Since $\int_0^4 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2} - 2t\right)\ln^2 t - \left(\frac{t^2}{2} - 4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^4 = 8\ln 4 - 12$
and
$\int_0^2 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2} - 2t\right)\ln^2 t - \left(\frac{t^2}{2} - 4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^2 = 6\ln 2 - 2\ln^2 2 - 7$
then $\frac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx + \int\limits_0^1 xv^2(x)\,dx\right) = \frac{1}{8}\left(8\ln 4 - 12 - 2\left(6\ln 2 - 2\ln^2 2 - 7\right)\right) =$
$\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}$.
Therefore, $4I = \ln^2 2 - \ln 2 + \frac{1}{2} - \left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}\right) = \frac{1}{2}\ln^2 2 - \frac{3}{2}\ln 2 + \frac{1}{4}$
$I = \frac{1}{8}\ln^2 2 - \frac{3}{8}\ln 2 + \frac{1}{16} \approx -0.137\,37$

Solution B.

Let $u(x) = \ln\left(\sqrt{1+x} + \sqrt{1-x}\right)$, $v(x) = \ln\left(\sqrt{1+x} - \sqrt{1-x}\right)$ and
$I = \int\limits_0^1 xu(x)\,v(x)\,dx.$

Calculation of $\int\limits_0^1 x\left(u^2(x) + v^2(x)\right)dx.$

1. Let $t = \sqrt{1+x} + \sqrt{1-x}$. Then $u^2(x) = \ln^2 t$ and
$t^2 = 2 + 2\sqrt{1-x^2} \iff \frac{t^2-2}{2} = \sqrt{1-x^2}$
yield $tdt = \frac{-xdx}{\sqrt{1-x^2}} \iff xdx = -\frac{t(t^2-2)}{2}dt.$
Hence, $\int\limits_0^1 xu^2(x)\,dx = -\int\limits_2^{\sqrt{2}} \frac{t(t^2-2)}{2}\ln^2 tdt = \frac{1}{2}\int\limits_{\sqrt{2}}^2 t\left(t^2-2\right)\ln^2 tdt;$

2. Let $t=\sqrt{1+x}-\sqrt{1-x}$. Then $v^2(x)=\ln^2 t$ and

$t^2=2-2\sqrt{1-x^2} \iff \dfrac{2-t^2}{2}=2\sqrt{1-x^2}$

yield $-tdt=\dfrac{-x}{\sqrt{1-x^2}}dx \iff xdx=\dfrac{t(2-t^2)}{2}dt$. Hence,

$$\int_0^1 xu^2(x)\,dx=\int_0^{\sqrt{2}}\frac{t(2-t^2)}{2}\ln^2 tdt=-\frac{1}{2}\int_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 tdt$$

and we obtain $\displaystyle\int_0^1 x\left(u^2(x)+v^2(x)\right)dx=\frac{1}{2}\int_{\sqrt{2}}^{2} t\left(t^2-2\right)\ln^2 tdt-\frac{1}{2}\int_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 tdt=$

$\displaystyle\frac{1}{2}\int_0^2 t\left(t^2-2\right)\ln^2 tdt-\int_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 tdt.$

Using integration by parts twice we obtain we obtain

$$\int t\left(t^2-2\right)\ln^2 tdt=\begin{bmatrix} p'=t^3-2t;\ p=\dfrac{t^4}{4}-t^2 \\ q=\ln^2 t;\ q'=\dfrac{2\ln t}{t}\end{bmatrix}=$$

$$\left(\frac{t^4}{4}-t^2\right)\ln^2 t-\int\left(\frac{t^3}{2}-2t\right)\ln tdt=$$

$$\left(\frac{t^4}{4}-t^2\right)\ln^2 t-\left(\frac{t^4}{8}-t^2\right)\ln t+\int\left(\frac{t^3}{8}-t\right)dt=$$

$$\left(\frac{t^4}{4}-t^2\right)\ln^2 t-\left(\frac{t^4}{8}-t^2\right)\ln t+\left(\frac{t^4}{32}-\frac{t^2}{2}\right).$$

Hence,

$$\int_0^2 t\left(t^2-2\right)\ln^2 tdt=\left(\left(\frac{t^4}{4}-t^2\right)\ln^2 t-\left(\frac{t^4}{8}-t^2\right)\ln t+\left(\frac{t^4}{32}-\frac{t^2}{2}\right)\right)_0^2=2\ln 2-\frac{3}{2},$$

$$\int_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 tdt=\left(\frac{\sqrt{2}^4}{4}-\sqrt{2}^2\right)\ln^2\sqrt{2}-\left(\frac{\sqrt{2}^4}{8}-\sqrt{2}^2\right)\ln\sqrt{2}+\left(\frac{\sqrt{2}^4}{32}-\sqrt{2}^2 2\right)=$$

$\dfrac{3}{4}\ln 2-\dfrac{1}{4}\ln^2 2-\dfrac{7}{8}$ and, therefore,

$$\int_0^1 x\left(u^2(x)+v^2(x)\right)dx=\frac{1}{2}\left(2\ln 2-\frac{3}{2}\right)-\left(\frac{3}{4}\ln 2-\frac{1}{4}\ln^2 2-\frac{7}{8}\right)=$$

$\frac{1}{4}\left(\ln^2 2+\ln 2+\frac{1}{2}\right).$

Since (using integration by parts again)
$\int\limits_0^1 x\ln^2(2x)\,dx=\frac{1}{4}\int\limits_0^1 2x\ln^2(2x)\cdot 2dx=\frac{1}{4}\int\limits_0^2 t\ln^2 t dt=\frac{1}{4}\left(\frac{t^2}{2}\left(\ln^2 t-\ln t+\frac{1}{2}\right)\right)_0^2=$
$\frac{1}{2}\left(\ln^2 2-\ln 2+\frac{1}{2}\right)$ then $I=\frac{1}{2}\left(\frac{1}{2}\left(\ln^2 2-\ln 2+\frac{1}{2}\right)-\frac{1}{4}\left(\ln^2 2+\ln 2+\frac{1}{2}\right)\right)=$
$\frac{1}{8}\left(\ln^2 2-3\ln 2+\frac{1}{2}\right)\approx -0.137\,37.$